2. 110Cinematica

Chapter 2

KINEMATICS

2.1 Introduction

This chapter will introduce the fundamental concepts for the study of the motion (kinematics) of robots. In the following kinematic analysis the constrained motion introduced by the joints will be generalized in order to embrace a larger class of mechanical systems. Free body, for example, will be considered as a particular case of “constrain”, thus allowing an easy and intuitive approach to describe a generic mechanical system.

After a summary of the necessary theoretical background, the chapter introduces the approach for describing a generic joint, and terminates with the resolution of the *direct kinematic problem* (*jacobians*).

2.2 Summary of general kinematics

2.2.1 *Vector derivatives*

The time <u>*derivative of the geometrical vector*</u> ρ, computed with respect to the frame $< a >$, is defined as the following geometrical vector:

$$\frac{d_a}{dt}\boldsymbol{\rho} \hat{=} \boldsymbol{i}_a \dot{x}_a + \boldsymbol{j}_a \dot{y}_a + \boldsymbol{k}_a \dot{z}_a \tag{2.1}$$

Similarly, for the same vector but in another frame $< b >$ we have:

$$\frac{d_b}{dt}\boldsymbol{\rho} \hat{=} \boldsymbol{i}_b \dot{x}_b + \boldsymbol{j}_b \dot{y}_b + \boldsymbol{k}_b \dot{z}_b \tag{2.2}$$

Let now find the relation between the 2.1 and 2.2:

$$\begin{aligned}\frac{d_a}{dt}\boldsymbol{\rho} &\hat{=} \frac{d_a}{dt}\left(\boldsymbol{i}_b x_b + \boldsymbol{j}_b y_b + \boldsymbol{k}_b z_b\right) = \\ &= \boldsymbol{i}_b \dot{x}_b + \boldsymbol{j}_b \dot{y}_b + \boldsymbol{k}_b \dot{z}_b + \left(\frac{d_a}{dt}\boldsymbol{i}_b\right)x_b + \left(\frac{d_a}{dt}\boldsymbol{j}_b\right)y_b + \left(\frac{d_a}{dt}\boldsymbol{k}_b\right)z_b = \\ &= \frac{d_b}{dt}\boldsymbol{\rho} + \left(\frac{d_a}{dt}\boldsymbol{i}_b\right)x_b + \left(\frac{d_a}{dt}\boldsymbol{j}_b\right)y_b + \left(\frac{d_a}{dt}\boldsymbol{k}_b\right)z_b\end{aligned} \tag{2.3}$$

Note that, in general, projecting the 2.1 on the frame $< b >$, we have:

$$^{b}\left(\frac{d_a}{dt}\boldsymbol{\rho}\right) \neq \frac{d_a}{dt}\,{}^{b}\boldsymbol{\rho} = \frac{d}{dt}\,{}^{b}\boldsymbol{\rho} \tag{2.4}$$

which defines the <u>derivative of algebraic vector</u>.

2.2.2 *Angular velocity*

Consider two frames in relative motion (Fig. 2.1). Since the matrix ${}^{a}_{b}R(t)$ is time-dependent, it is possible to define as *angular velocity of the frame* $< b >$ *w.r.t. the frame* $< a >$ the vector $\boldsymbol{\omega}_{b/a}$, which, at any time instant, provides the following information:

a) its versor indicates the axis around which, at any time instant, an observer sitting in $< a >$ can suppose that $< b >$ is rotating w.r.t. $< a >$;

b) the component along its versor indicates the effective instantaneous angular velocity (measured in *radians/sec.*)

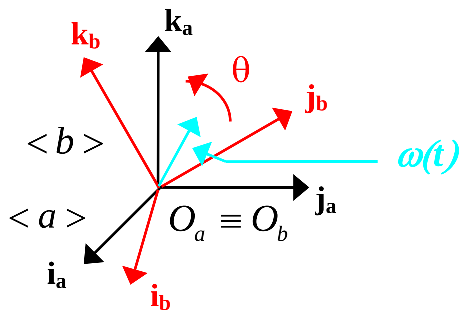

Fig. 2.1

We can associate to the angular velocity vector the following differential form:

$$d\boldsymbol{\theta}_{b/a} = \boldsymbol{\omega}_{b/a} dt \tag{2.5}$$

The 2.5, however, does not coincide, in general, with any exact differential[1].

We can now rewrite the 2.3 using the vector $\boldsymbol{\omega}_{b/a}$; to that end we need the *Poisson formulas,* which we will not proof:

$$\left| \begin{array}{l} \frac{d_a}{dt}\boldsymbol{i}_b = \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{i}_b \\ \frac{d_a}{dt}\boldsymbol{j}_b = \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{j}_b \\ \frac{d_a}{dt}\boldsymbol{k}_b = \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{k}_b \end{array} \right. \tag{2.6}$$

Hence, the 2.3 becomes:

$$\frac{d_a}{dt}\boldsymbol{\rho} = \frac{d_b}{dt}\boldsymbol{\rho} + \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{\rho} \tag{2.7}$$

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[1] In mathematics, a differential dQ is said to be exact, as contrasted with an inexact differential, if the differentiable function Q exists.

Note that, if the module of the vector $\boldsymbol{\rho}$ is constant and $\boldsymbol{\rho}$ is integral with $< b >$, the 2.7 becomes:

$$\frac{d_a}{dt}\boldsymbol{\rho} = \boldsymbol{\omega}_{b/a} \wedge \boldsymbol{\rho} \tag{2.8}$$

The above relation is very useful in case of rigid bodies.

Let’s check out now some ***properties of the angular velocity vector***.

1) $\boldsymbol{\omega}_{b/a} = -\boldsymbol{\omega}_{a/b}$ (2.9)

2) given n frames, the angular velocity $\boldsymbol{\omega}_{k/h}$ of the a generic k with respect to any other h can be obtained by adding (vector sum) the successive angular velocities encountered in any path between k e h.

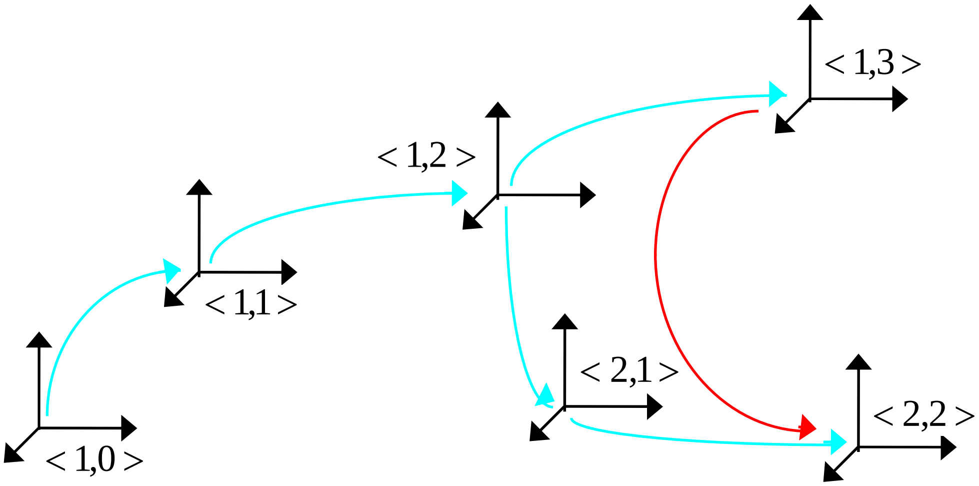

Fig. 2.2

In the example of fig. 2.2 we have:

$$\boldsymbol{\omega}_{(2,2)/(1,3)} = \boldsymbol{\omega}_{(2,2)/(2,1)} + \boldsymbol{\omega}_{(2,1)/(1,2)} + \boldsymbol{\omega}_{(1,2)/(1,3)}$$

2.2.3 *Time derivatives for points in the space*

Consider now two reference systems $< a >$ and $< b >$, with the latter moving relatively to the former, with any general (either rotational and translational) motion.

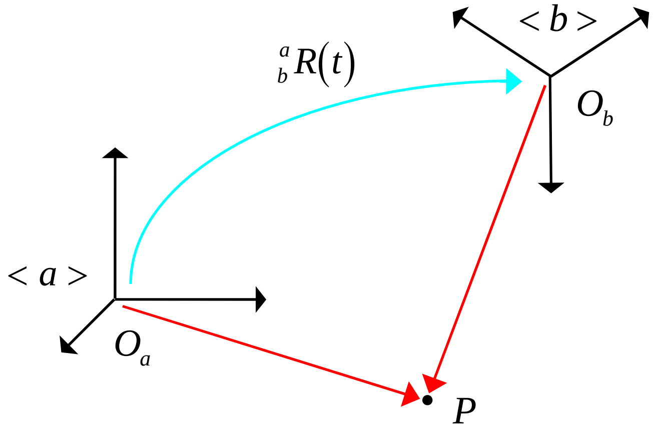

Fig. 2.3

We can define *velocity of one point P* , computed with respect to the frame $<a>$, the geometrical vector:

$$\boldsymbol{v}_{p/a} = \frac{d_a}{dt}(P - O_a) = \boldsymbol{i}_a\dot{x}_{p/a} + \boldsymbol{j}_a\dot{y}_{p/a} + \boldsymbol{k}_a\dot{z}_{p/a} \tag{2.10}$$

Equally:

$$\boldsymbol{v}_{p/b} = \frac{d_b}{dt}(P - O_b) \tag{2.11}$$

Let's now find the relation between the 2.10 and the 2.11:

$$\boldsymbol{v}_{p/a} = \frac{d_a}{dt}(P - O_a) = \frac{d_a}{dt}(O_b - O_a) + \frac{d_a}{dt}(P - O_b)$$

Indicating with $\boldsymbol{v}_{b/a}$ the velocity of the origin of the frame $<b>$ with respect to the $<a>$, the previous relation becomes:

$$\boldsymbol{v}_{p/a} = \boldsymbol{v}_{b/a} + \frac{d_a}{dt}(P - O_b) \tag{2.12}$$

and, by means of the 2.7 (with the opportune indexes):

$$\boldsymbol{v}_{p/a} = \boldsymbol{v}_{b/a} + \frac{d_b}{dt}(P - O_b) + \boldsymbol{\omega}_{b/a} \wedge (P - O_b)$$

that is:

$$\boldsymbol{v}_{p/a} = \boldsymbol{v}_{b/a} + \boldsymbol{v}_{P/b} + \boldsymbol{\omega}_{b/a} \wedge \left(P - O_b\right) \tag{2.13}$$

In the frequent case where P is integral with $< b >$, it happens to be $\boldsymbol{v}_{P/b} = 0$. Hence:

$$\boldsymbol{v}_{p/a} = \boldsymbol{v}_{b/a} + \boldsymbol{\omega}_{b/a} \wedge \left(P - O_b\right) \tag{2.14}$$

2.2.4 *Generalized velocity*

In order to completely describe the relative motion between two reference frames (see fig. 2.4) it is possible to organize in one vector the angular velocity and the velocity of the origin of the second frame with respect to the first.

For example, in order to specify *how* the frame $< b >$ moves with respect to $< a >$, let’s introduce the vector:

$$\dot{\boldsymbol{X}}_{b/a} \hat{=} \begin{bmatrix} \boldsymbol{\omega}_{b/a} \\ \boldsymbol{v}_{b/a} \end{bmatrix} \tag{2.15}$$

where $\boldsymbol{v}_{b/a}$ is the velocity of the origin of the frame $< b >$ with respect to $< a >$ (velocity of the vector $O_b - O_a$).

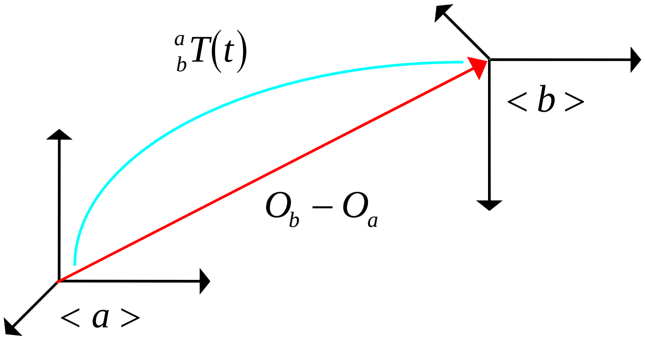

Fig. 2.4

The vector $\dot{\boldsymbol{X}}_{b/a}$ is named *generalized velocity* of the frame $<b>$ with respect to the frame $<a>$. It can be projected on any frame; if, for example, the this is the base frame $<0>$, we have:

$$ {}^{0}\dot{\boldsymbol{X}}_{b/a} \hat{=} \begin{bmatrix} {}^{0}\boldsymbol{\omega}_{b/a} \\ {}^{0}\boldsymbol{v}_{b/a} \end{bmatrix} $$

Of course, the above relation holds even for a generic point integral with the frame *b*. Let *P* be a point belonging to $<b>$; we define generalized velocity of *P* with respect to $<a>$ the quantity:

$$ \dot{\boldsymbol{X}}_{P/a} \hat{=} \begin{bmatrix} \boldsymbol{\omega}_{b/a} \\ \boldsymbol{v}_{P/a} \end{bmatrix} \tag{2.16} $$

with $\boldsymbol{v}_{P/a}$ defined by 2.14.

2.2.5 *Derivative of the rotation matrix*

With respect to figure 2.1, we want now find a relation between the derivative ${}^{a}_{b}\dot{R}$ of the orientation matrix and the angular velocity vector $\boldsymbol{\omega}_{b/a}$. Recalling the 1.14:

$$ {}^{a}_{b}R = \left[{}^{a}\boldsymbol{i}_{b} \quad {}^{a}\boldsymbol{j}_{b} \quad {}^{a}\boldsymbol{k}_{b} \right] \tag{2.17} $$

and deriving with respect the time we have:

$$ {}^{a}_{b}\dot{R} = \left[\frac{d}{dt}\,{}^{a}\boldsymbol{i}_{b} \quad \frac{d}{dt}\,{}^{a}\boldsymbol{j}_{b} \quad \frac{d}{dt}\,{}^{a}\boldsymbol{k}_{b} \right] = $$

$$ = \left[{}^{a}\left(\frac{d_{a}}{dt}\boldsymbol{i}_{b}\right) \quad {}^{a}\left(\frac{d_{a}}{dt}\boldsymbol{j}_{b}\right) \quad {}^{a}\left(\frac{d_{a}}{dt}\boldsymbol{k}_{b}\right) \right] = $$

$$ = \left[{}^{a}\left(\boldsymbol{\omega}_{b/a} \wedge \boldsymbol{i}_{b}\right) \quad {}^{a}\left(\boldsymbol{\omega}_{b/a} \wedge \boldsymbol{j}_{b}\right) \quad {}^{a}\left(\boldsymbol{\omega}_{b/a} \wedge \boldsymbol{k}_{b}\right) \right] = $$

$$ = \left[\left[{}^{a}\boldsymbol{\omega}_{b/a} \wedge\right] {}^{a}\boldsymbol{i}_{b} \quad \left[{}^{a}\boldsymbol{\omega}_{b/a} \wedge\right] {}^{a}\boldsymbol{j}_{b} \quad \left[{}^{a}\boldsymbol{\omega}_{b/a} \wedge\right] {}^{a}\boldsymbol{k}_{b} \right] = $$

$$ = \left[{}^{a}\boldsymbol{\omega}_{b/a} \wedge\right] \left[{}^{a}\boldsymbol{i}_{b} \quad {}^{a}\boldsymbol{j}_{b} \quad {}^{a}\boldsymbol{k}_{b} \right] $$

that is:

$$ {}^{a}_{b}\dot{R} = \left[{}^{a}\boldsymbol{\omega}_{b/a} \wedge\right] {}^{a}_{b}R \tag{2.18} $$

Substituting the 1.25 within the previous 2.18 we obtain a dual formula:

$$ {}_{b}^{a}\dot{R} = {}_{b}^{a}R\left[{}^{b}\boldsymbol{\omega}_{b/a} \wedge\right] \tag{2.19}$$

The 2.11 and 2.12 are helpful for numerically evaluating the time evolution of ${}_{b}^{a}R$; precisely:

$$ {}_{b}^{a}R(t+dt) = {}_{b}^{a}R(t) + {}_{b}^{a}\dot{R}\,dt \tag{2.20}$$

2.3 Joint kinematics

In general, the set of all the relative movements between two unconstrained rigid bodies forms a group, G, consisting of all rotations and translations of $\Re^3$. A generic element of G may be represented by a matrix (see fig. 2.4):

$$ {}_{b}^{a}T = \begin{bmatrix} {}_{b}^{a}R & L \\ 0 & 1 \end{bmatrix} \in \mathsf{G}, \quad {}_{b}^{a}R \in SO(3), \quad L \in \Re^3 \tag{2.21}$$

where $SO(3)$ is the group of the rotation matrices.

The group G is also said *Special Euclidean group* $SE(3)$. Its tangent space is isomorphic to G×*g*, where *g* is the spaceof the generalized velocities (Lie algebra). A generic element of belonging to *g* can be formally represented using the 2.19:

$$ {}^{b}\tilde{X}_{b/a} = {}_{b}^{a}T^{-1}\,{}_{b}^{a}\dot{T} = \begin{bmatrix} {}_{b}^{a}R^{T} & -L \\ 0 & 1 \end{bmatrix}\begin{bmatrix} {}_{b}^{a}\dot{R} & \dot{L} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} {}_{b}^{a}R^{T}\,{}_{b}^{a}\dot{R} & {}_{b}^{a}R^{T}\dot{L} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \left[{}^{b}\boldsymbol{\omega}_{b/a} \wedge\right] & {}^{b}\boldsymbol{v}_{b/a} \\ 0 & 0 \end{bmatrix} \tag{2.22}$$

The element ${}^{b}\tilde{X}_{b/a}$ is the re-organization of the generalized velocity ${}^{b}\dot{X}_{b/a}$ into a particular matrix form.

The joint, already introduced in the paragraph 1.4, can be represented through a constrain relation in the tangent space G×*g*, involving the generalized velocity of the frame $<b>$ with respect to $<a>$:

$$ A(\boldsymbol{q})\,{}^{b}\dot{X}_{b/a} = 0 \tag{2.23}$$

where $\boldsymbol{q}$ is the configuration. Joints of this type are known as *kinematic*, since they constrain the relative velocity.

The 2.23 means that $\dot{X}$ belongs to the kernel of a A:

$$^{b}\dot{X}_{b/a} \in \Delta(q) = \ker\left[A(\boldsymbol{q})\right] \tag{2.24}$$

and, if the distribution $\Delta(\boldsymbol{q})$ is integrable, the constrain introduced by the joint is said Holonomic.

In case the axis are integral with at least one of the two bodies, the matrix A is constant (independent of the configuration). This class of joints is called *Simple Kinematic Joints*. In this case, let's introduce a matrix H, which columns form a basis of the kernel of A. All the solutions of the 2.23 are:

$$^{b}\dot{X}_{b/a} = Hp, \quad p \in \Re^{r} \tag{2.25}$$

with $r = \dim(\ker[A])$, degree of freedom of the joint.

H is the *Joint Matrix*, while p is often named *quasivelocity*.

Figure 2.5 shows some examples of joint matrices associated to commonly used joints.

$\begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}$	$\begin{bmatrix}0\\0\\1\\0\\0\\k\end{bmatrix}$	$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&0\end{bmatrix}$
1 DOF Revolute joint (body z-axis)	1 DOF Prismatic joint (body x-axis)	1 DOF Screw joint (body z-axis)	3 dof Spherical joint

Fig. 2.5

2.3.1 *Parameterization of simple kinematic joints*

The configuration of the joint is, in general, defined by the differential equation 2.22:

$${}^{a}_{b}\dot{T} = {}^{a}_{b}T\, {}^{b}\tilde{X}_{b/a} \tag{2.26}$$

The last can be re-written using the 2.25 in the following way:

$$ {}_b^a\dot{T} = {}_b^aT \begin{bmatrix} [H_1 p \wedge] & H_2 p \\ 0 & 0 \end{bmatrix} \tag{2.27}$$

where H_1 contains the first three rows of H and H_2 the last three. Once the quasivelocities p are known, equation 2.27 can be integrated to compute the time evolution of the transformation matrix $T(t)$.

However, equation 2.27 gives more information than necessary, since the movement is constrained to lie on a subgroup of G.

For example, let's consider a one degree of freedom jount (r=1). In this case the joint matrix $H = h \in \Re^6$ is made of only one column and the transformation matrix T can be parameterized using one parameter q_1:

$$ {}_b^aT(q_1) = \begin{bmatrix} R(q_1) & L(q_1) \\ 0 & 1 \end{bmatrix} \tag{2.28}$$

where:

$$R(q_1) = e^{[h_1 \wedge] q_1} \tag{2.29}$$

$$L(q_1) = \int_0^{q_1} e^{[h_1 \wedge]\sigma} h_2 \, d\sigma \tag{2.30}$$

with $h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix}$.

Equation 2.29 can be easily derived by considering that vector h_1 represents the direction of the rotation axis of the joint in the frame $< b >$ and recalling the exponential representation of rotations 1.23.

Similarly, h_2 is the direction of the translation axis of the joint in the reference frame $< b >$. Therefore we can write:

$$\frac{dL}{dq_1} = R(q_1) h_2 \tag{2.31}$$

that, once integrated, results in the equation 2.30.

In case H is made $r > 1$ columns, if the joint is holonomic (and only in this case) its parameterization can be derived with the following procedure:

1) associate every column to a joint variable

2) Compute the transformation matrix for each column according to the 2.28,2.29,2.30
3) Compute the final transformation matrix by multiplying the r matrices as follows:

$$T(\boldsymbol{q}) = T_r(q_r)\ldots T_2(q_2)\,T_1(q_1) \tag{2.32}$$

2.3.2 *Example: spherical joint*

The spherical joint is ccharacterized by a three-columns joint matix:

$$H_{sp} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.33}$$

Let $\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$. From equation 2.25 we have:

$$^{b}\dot{X}_{b/a} = H\boldsymbol{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

This means that the frame $< b >$ can move relatively to $< a >$ with any rotation around its coordinate axis.

The parameterization can be obtained by first applying equations 2.29 e 2.30 to every column:

$$R_1(q_1) = e^{\left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\wedge\right] q_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(q_1) & -sin(q_1) \\ 0 & sin(q_1) & \cos(q_1) \end{bmatrix}$$

$$L_1(q_1) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2(q_2) = e^{\left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^{\wedge} q_2\right]} = \begin{bmatrix} \cos(q_2) & 0 & sin(q_2) \\ 0 & 1 & 0 \\ -sin(q_2) & 0 & \cos(q_2) \end{bmatrix}$$

$$L_2(q_2) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_3(q_3) = e^{\left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}^{\wedge} q_3\right]} = \begin{bmatrix} \cos(q_3) & -sin(q_3) & 0 \\ sin(q_3) & \cos(q_3) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$L_3(q_3) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Finally, the transformation (configuration) matrix of the joint can be computed using the 2.32 (see also figure 2.6):

$${}^{a}_{b}T(\boldsymbol{q}) = T_3(q_3)\,T_2(q_2)\,T_1(q_1) =$$

$$\begin{bmatrix} \cos(q3)\cos(q2) & \cos(q3)\sin(q2)\sin(q1) - \sin(q3)\cos(q1) & \cos(q3)\sin(q2)\cos(q1) + \sin(q3)\sin(q1) & 0 \\ \sin(q3)\cos(q2) & \sin(q3)\sin(q2)\sin(q1) + \cos(q3)\cos(q1) & \sin(q3)\sin(q2)\cos(q1) - \cos(q3)\sin(q1) & 0 \\ -\sin(q2) & \cos(q2)\sin(q1) & \cos(q2)\cos(q1) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.34}$$

where, according to the 2.28:

$$T_i(q_i) = \begin{bmatrix} R_i(q_i) & L_i(q_i) \\ 0 & 1 \end{bmatrix}$$

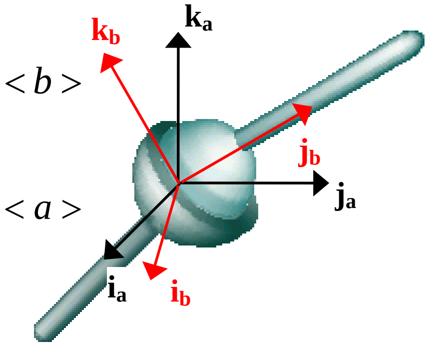

Fig. 2.6

The rotational part of the 2.34 coincides with the 1.32. As a matter of fact, with this choice, the parameters representing the joint are the Roll, Pitch and Yaw angles.

2.3.3 *Example: screw joint*

With the joint matrix, it is possible to associate to the same joint variable either a rotational and translational axis. Let's consider, for example, a joint matrix *H* like the following:

$$H_{sc} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \tag{2.35}$$

A variation of q_1 results in a rotation around the ***i***–axis and, in the same time, a translation along the same axis, similarly to the movement of a screw.

The associated matrices are:

$$R_1(q_1) = e^{\left[\begin{bmatrix}1\\0\\0\end{bmatrix}\wedge\right] q_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(q_1) & -sin(q_1) \\ 0 & sin(q_1) & \cos(q_1) \end{bmatrix}$$

$$L_{sc}(q_1) = \int_0^{q_1} R_{sc}(\sigma)\begin{bmatrix}1\\0\\0\end{bmatrix} d\sigma = \int_0^{q_1}\begin{bmatrix}1\\0\\0\end{bmatrix} d\sigma = \begin{bmatrix}q_1\\0\\0\end{bmatrix}.$$

Hence:

$$T_{sc}(q_1) = \begin{bmatrix}1 & 0 & 0 & q_1\\ 0 & \cos(q_1) & -sin(q_1) & 0\\ 0 & sin(q_1) & \cos(q_1) & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{2.36}$$

2.3.4 *Kinematic equation of simple joints*

We now want to find a relation between the quasivelocities and the time derivative of the variables used to parameterize the joint.

Let's consider a simple joint described by the matrix:

$$H = \begin{bmatrix}\boldsymbol{h}_{11} & \cdots & \boldsymbol{h}_{1r}\\ \boldsymbol{h}_{21} & \cdots & \boldsymbol{h}_{2r}\end{bmatrix} \in \Re^{6\times r}$$

parameterized as in par. 2.3.1. We define *kinematic equation* the following relation:

$$\dot{\boldsymbol{q}} = \Gamma(\boldsymbol{q})\,\boldsymbol{p} \tag{2.37}$$

where the matrix $\Gamma(\boldsymbol{q})$ is defined with the following recursive algorithm:

1) For $j = 1..r$ let the matrices R_j e L_j defined as:

$$\mathsf{R}_j(q_j \ldots q_1) = R_j(q_j)\mathsf{R}_{j-1}(q_{j-1}\ldots q_1), \mathsf{R}(0) = I \tag{2.38}$$

$$\mathsf{L}_j(q_j \ldots q_1) = R_j(q_j)\mathsf{L}_{j-1}(q_{j-1}\ldots q_1) + L_j(q_j), \mathsf{L}(0) = I \tag{2.39}$$

2) Let B defined as:

$$B(\mathbf{q}) = \begin{bmatrix}\mathbf{b}_{11} & \cdots & \mathbf{b}_{1r}\\ \mathbf{b}_{21} & \ldots & \mathbf{b}_{2r}\end{bmatrix} \tag{2.40}$$

where:

$$[\mathbf{b}_{1i}\wedge] = \mathsf{R}_{\text{i-1}}{}^{T}[\mathbf{h}_{1i}\wedge]\mathsf{R}_{\text{i-1}} \tag{2.41}$$

$$\mathbf{b}_{2i} = \mathsf{R}_{\text{i-1}}{}^{T}[\mathbf{h}_{1i}\wedge]\mathsf{L}_{\text{i-1}} + \mathsf{R}_{\text{i-1}}\mathbf{h}_{2i} \tag{2.42}$$

3) Compute $\Gamma(\boldsymbol{q})$ as:

$$\Gamma(\boldsymbol{q}) = B^*(\boldsymbol{q})\,H$$

where $B^*(\boldsymbol{q})$ denotes a right-inverse of $B(\boldsymbol{q})$.

<u>Proof.</u>

Let's compute the derivative of the transformation matrix as a function of the derivatives $\dot{\boldsymbol{q}}$:

$$ {}^a_b\dot{T} = \sum_{i=1}^{r} \frac{\partial\, {}^a_bT(\mathbf{q})}{\partial q_1}\,\dot{q}_1 = \sum_{i=1}^{r}\left[T_r(q_r)\ldots T_{i+1}(q_{i+1})\frac{\partial T_i(q_i)}{\partial q_i}T_{i-1}(q_{i-1})\ldots T_1(q_1)\right]\dot{q}_i \tag{2.43}$$

Pre-multiplying by ${}^a_bT(\boldsymbol{q})^{-1} = \left\{T_r(q_r)\ldots T_2(q_2)\,T_1(q_1)\right\}^{-1}$ and considering the equation 2.22 we obtain:

$$\begin{aligned}
&{}^a_bT(\boldsymbol{q})^{-1}\;{}^a_b\dot{T} = {}^b\tilde{X}_{b/a} = \\
&= \sum_{i=1}^{r} T_1(q_1)^{-1}\,T_2(q_2)^{-1}\ldots T_r(q_r)^{-1}\left[T_r(q_r)\ldots T_{i+1}(q_{i+1})\frac{\partial T_i(q_i)}{\partial q_i}T_{i-1}(q_{i-1})\ldots T_1(q_1)\right]\dot{q}_i = \\
&= \sum_{i=1}^{r}\left[T_1(q_1)^{-1}\,T_2(q_2)^{-1}\ldots T_i(q_i)^{-1}\,\frac{\partial T_i(q_i)}{\partial q_i}T_{i-1}(q_{i-1})\ldots T_1(q_1)\right]\dot{q}_i = \\
&= \sum_{i=1}^{r}\left[T_{i-1}(q_{i-1})\ldots T_1(q_1)\right]^{-1} T_i(q_i)^{-1}\,\frac{\partial T_i(q_i)}{\partial q_i}T_{i-1}(q_{i-1})\ldots T_1(q_1)\,\dot{q}_i
\end{aligned} \tag{2.44}$$

From equation 2.27 it is possible to show that:

$$T_i(q_i)^{-1}\,\frac{\partial T_i(q_i)}{\partial q_i} = \begin{bmatrix} [\boldsymbol{h}_{1i}\wedge] & \boldsymbol{h}_{2i} \\ 0 & 0 \end{bmatrix} \tag{2.45}$$

Then, defining the matrix

$$U_j(q_j\ldots q_1) = T_j(q_j)\ldots T_1(q_1)$$

the 2.44 becomes:

$${}^b\tilde{X}_{b/a} = \sum_{i=1}^{r} U_{i-1}{}^{-1}\begin{bmatrix} [\boldsymbol{h}_{1i}\wedge] & \boldsymbol{h}_{2i} \\ 0 & 0 \end{bmatrix} U_{i-1}\,\dot{q}_i \tag{2.46}$$

It is easy to show that the matrix U_j has the form:

$$U_j = \begin{bmatrix} \mathsf{R}_j\left(q_j \ldots q_1\right) & \mathsf{L}_j\left(q_j \ldots q_1\right) \\ 0 & 1 \end{bmatrix} \tag{2.47}$$

with R e L defined as in 2.38 e 2.39. Hence, the 2.46 becomes:

$$\tilde{X} = \sum_{i=1}^{r} \begin{bmatrix} \mathsf{R}_{i-1}{}^{T}\left[\boldsymbol{h}_{1i} \wedge\right]\mathsf{R}_{i-1} & \mathsf{R}_{i-1}{}^{T}\left[\boldsymbol{h}_{1i} \wedge\right]\mathsf{L}_{i-1} + \mathsf{R}_{i-1}\boldsymbol{h}_{2i} \\ 0 & 0 \end{bmatrix} \dot{q}_i \tag{2.48}$$

As already seen in the 1.25, the matrix $\mathsf{R}_{i-1}{}^{T}\left[\boldsymbol{h}_{1i} \wedge\right]\mathsf{R}_{i-1}$ is still anti-symmetric. Is then possible to define a vector $\boldsymbol{b}_{1i} \in \Re^3$ such as:

$$\left[\boldsymbol{b}_{1i} \wedge\right] = \mathsf{R}_{i-1}{}^{T}\left[\boldsymbol{h}_{1i} \wedge\right]\mathsf{R}_{i-1} \tag{2.49}$$

Let’s define also the vector:

$$\boldsymbol{b}_{1i} = \mathsf{R}_{i-1}{}^{T}\left[\boldsymbol{h}_{1i} \wedge\right]\mathsf{L}_{i-1} + \mathsf{R}_{i-1}{}^{T}\,\boldsymbol{h}_{2i} \tag{2.50}$$

Finally the 2.48 becomes:

$${}^{b}\tilde{X}_{b/a} = \sum_{i=1}^{r} \begin{bmatrix} \left[\boldsymbol{b}_{1i} \wedge\right] & \boldsymbol{b}_{2i} \\ 0 & 0 \end{bmatrix} \dot{q}_i \tag{2.51}$$

It is possible to reorganize the 2.51 in a compact form, substituting the anti-symmetric matrices with the relative vectors:

$${}^{b}\dot{X}_{b/a} = \sum_{i=1}^{r} \begin{bmatrix} \boldsymbol{b}_{1i} \\ \boldsymbol{b}_{2i} \end{bmatrix} \dot{q}_i = B(\boldsymbol{q})\,\dot{\boldsymbol{q}} \tag{2.52}$$

with $B(\boldsymbol{q})$ defined as in 2.40, 2.41 e 2.42. Recalling the 2.25 we then have:

$$B(\boldsymbol{q})\,\dot{\boldsymbol{q}} = H\boldsymbol{p} \tag{2.53}$$

Let $B(\boldsymbol{q})^{*}$ be a left inverse of $B(\boldsymbol{q})$; it certainly exists in a neighborhood of $\boldsymbol{q} = \boldsymbol{0}$, since $B(\boldsymbol{0}) = H$ and H, as already seen with the equation 2.25, is full rank, since its columns form a basis for the kernel of A. Hence we obtain:

$$\dot{\boldsymbol{q}} = \left[B(\boldsymbol{q})^{*}\,H\right]\boldsymbol{p} \tag{2.54}$$

and:

$$\Gamma(\boldsymbol{q}) = B^{*}(\boldsymbol{q})\,H \tag{2.55}$$

2.3.5 *Compound joints*

Not all the joints can be considered simple. In several cases, its action must be regarded as a *sequence* of simple joints, as for example the joint with roll-pitch-yaw angles of figure 2.7. Joints of this model will be named *compound*.

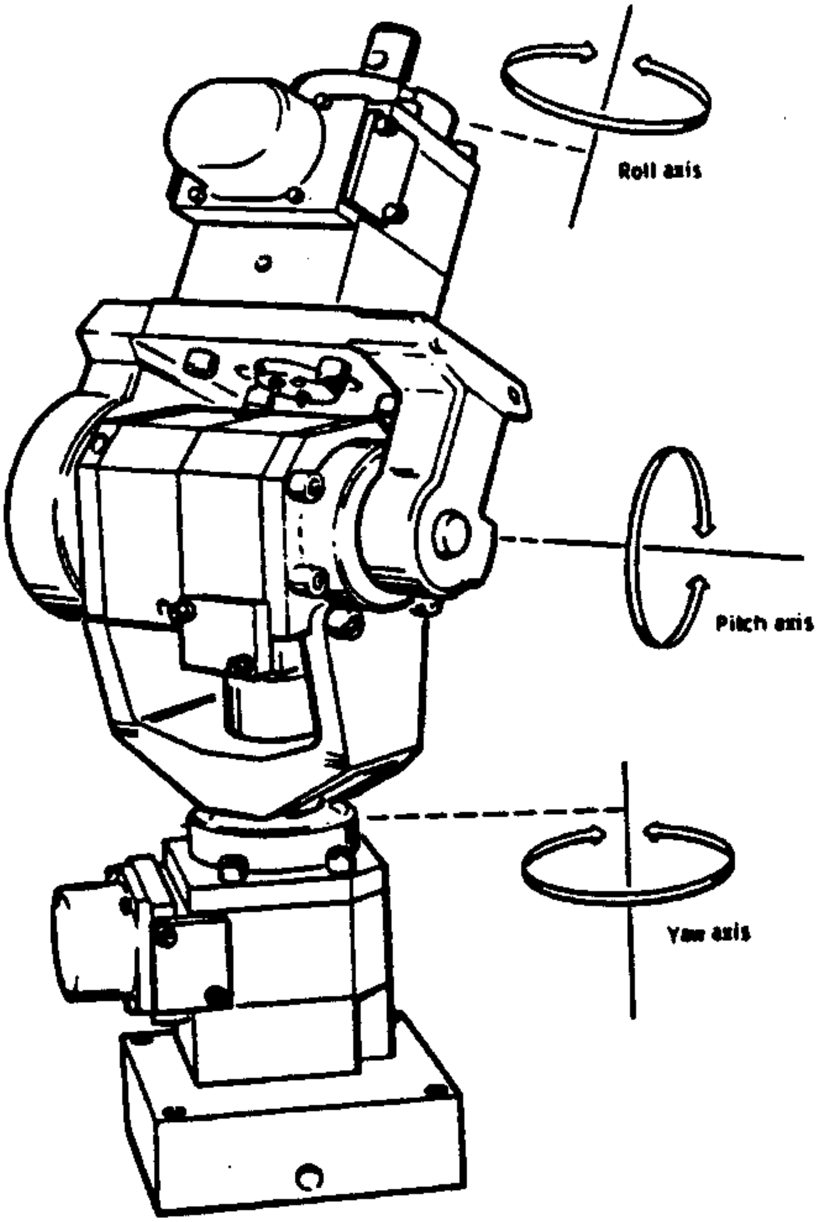

Fig. 2.7

In a more general view, a compound joint can be characterized by the relative motion of a sequence of *k* frames such as the relative motion between frames is constrained by a simple joint.

Each of the ***k* simple joints** is characterized by:

- a joint matrix H_i having r_i columns[2];
- a vector of parameters $\boldsymbol{q}_i$ of dimension r_i;
- a quasivelocity vector $\boldsymbol{p}_i$ of dimension r_i;
- a kinematic matrix Γ_i of dimension $r_i \times r_i$

In this way, it is possible to consider the *compound joint* as one unique “entity” having:

- a configuration vector:

$$\boldsymbol{q} = \begin{bmatrix} \boldsymbol{q}_1 \\ \vdots \\ \boldsymbol{q}_k \end{bmatrix}$$

- a quasivelocity vector:

$$\boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}_1 \\ \vdots \\ \boldsymbol{p}_k \end{bmatrix}$$

- a kinematic matrix such as:

$$\dot{\boldsymbol{q}} = \begin{bmatrix} \Gamma_1(\boldsymbol{q}_1) & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \Gamma_2(\boldsymbol{q}_2) & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \Gamma_k(\boldsymbol{q}_k) \end{bmatrix} \boldsymbol{p}$$

- a configuration matrix $T(\boldsymbol{q}) = T_k(\boldsymbol{q}_i)\,T_{k-1}(\boldsymbol{q}_{k-1})\ldots T_1(\boldsymbol{q}_1)$, where we assume that the (simple) joins number 1 is the outer one.

In order to compute the global joint matrix, let’s consider the simplified case with $r_i = 1, \forall \mathrm{i}$. We can then determine the global joint matrix with the

---

[2] In RDS r_i is always 1, that is only compound joints made of successions of one DOF simple joints can be described.

following recursive procedure, similar to the one used for computing the kinematic matrix of the simple joints.

Let’s consider a compound joint made of *k* simple joints, each one of one degree of freedom and having a joint matrix $\mathsf{h}_i = \begin{bmatrix} \mathsf{h}_{1i} \\ \mathsf{h}_{2i} \end{bmatrix}$. Then, the global joint matrix *H* is given by:

$$H(\boldsymbol{q}) = \begin{bmatrix} \boldsymbol{h}_{11} & \cdots & \boldsymbol{h}_{1r} \\ \boldsymbol{h}_{21} & \cdots & \boldsymbol{h}_{2r} \end{bmatrix} \tag{2.56}$$

where:

$$\left[\boldsymbol{h}_{1i} \wedge\right] = \mathsf{R}_{\mathsf{i-1}}{}^{T} \left[\mathsf{h}_{1i} \wedge\right] \mathsf{R}_{\mathsf{i-1}} \tag{2.57}$$

$$\boldsymbol{h}_{2i} = \mathsf{R}_{\mathsf{i-1}}{}^{T} \left[\mathsf{h}_{1i} \wedge\right] \mathsf{L}_{\mathsf{i-1}} + \mathsf{R}_{\mathsf{i-1}}{}^{T}\, \mathsf{h}_{2i} \tag{2.58}$$

and:

$$\mathsf{R}_j\left(q_j \ldots q_1\right) = R_j\left(q_j\right) \mathsf{R}_{j-1}\left(q_{j-1} \ldots q_1\right), \quad \mathsf{R}(0) = I \tag{2.59}$$

$$\mathsf{L}_j\left(q_j \ldots q_1\right) = R_j\left(q_j\right) \mathsf{L}_{j-1}\left(q_{j-1} \ldots q_1\right) + L_j\left(q_j\right), \quad \mathsf{L}(0) = I \tag{2.60}$$

The proof will not be reported.

In conclusion, similarly to the simple joints, a compound joint can be described by a joint matrix *H* such as the 2.25 still holds. That is:

$${}^{b}\dot{X}_{b/a} = H\boldsymbol{p}, \quad \boldsymbol{p} \in \Re^{r} \tag{2.61}$$

In this case (and in the hypothesis that $r_i = 1, \forall \mathrm{i}$) the quasivelocities coincide with the derivatives of the joint variables, and the matrix *H* performs a projection, in the outer frame, of the velocities at the single component joints.

2.3.6 *Example: joint with Roll-Pitch-Yaw angles.*

Let’s consider a joint made of a succession of three one-DOF, rotational simple joints, which rotational directions are respectively the axis ***i***, ***j***, ***k*** (starting from the outer frame). Organizing the joint matrices into a unique matrix we have:

$$\mathrm{H} = \begin{bmatrix} \mathrm{h}_1 & \mathrm{h}_2 & \mathrm{h}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.62}$$

The computation of *H* using the 2.(56÷60) gives the following result:

$$H = \begin{bmatrix} 1 & 0 & -\sin(q_2) \\ 0 & \cos(q_1) & \sin(q_1)\cos(q_2) \\ 0 & -\sin(q_1) & \cos(q_1)\cos(q_2) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.63}$$

while the configuration matrix is still given by the 2.34.

The application of equation 2.62 gives:

$$^{b}\dot{X}_{b/a} = H\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2\cos(q_1) + p_3\sin(q_1)\cos(q_2) \\ -p_2\sin(q_1) + p_3\cos(q_1)\cos(q_2) \\ 0 \\ 0 \\ 0 \end{bmatrix}^{r} \tag{2.64}$$

where it is easy to verify that the first halfis the sum of the projections of the three angular velocities on the outer frame $< b >$.

2.4 Kinematics of robotic systems

We have now all the rudiments to solve the *direct kinematic problem* of the robotic systems, i.e. to compute the generalized velocity of every point of the structure for a particular value of the vector ***p***.

To simplify the study let's consider first a linear chain (one branch only, indicating with $<0>$ the base frame), made of k links e k joints, each one characterized by:

- r_i degrees of freedom;
- Joint matrix $H_i \in \Re^{6\times r_i}$;
- configuration vector $\boldsymbol{q}_i \in \Re^{r_i}$
- quasivelocity vector $\boldsymbol{p}_i \in \Re^{r_i}$

Let c_i be a point of a generic link *i* and ${}^i\dot{\boldsymbol{X}}_{i/0}$ its generalized velocity w.r.t. the base frame as defined in 2.16, projected on the frame of the same link; relatively to the generalized velocity of the origin O_i of the frame <*i*> we have:

$$
{}^i\dot{\boldsymbol{X}}_{c_i/0} = \begin{bmatrix} {}^i\boldsymbol{\omega}_{c_i/0} \\ {}^i\boldsymbol{v}_{c_i/0} \end{bmatrix} = \begin{bmatrix} {}^i\boldsymbol{\omega}_{i/0} \\ {}^i\boldsymbol{v}_{i/0} + {}^i\boldsymbol{\omega}_{i/0} \Lambda\, {}^i\boldsymbol{r}_{O_ic_i} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\left[{}^i\boldsymbol{r}_{O_ic_i} \Lambda\right] & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} {}^i\boldsymbol{\omega}_{i/0} \\ {}^i\boldsymbol{v}_{i/0} \end{bmatrix} =
$$

$$
= \boldsymbol{\phi}({}^i\boldsymbol{r}_{O_ic_i})\, {}^i\dot{\boldsymbol{X}}_{c_i/0} \tag{2.65}
$$

where:

$$
\boldsymbol{\phi}(\boldsymbol{r}) = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -\left[\boldsymbol{r}\Lambda\right] & \boldsymbol{I} \end{bmatrix}, \tag{2.66}
$$

$$
{}^i\boldsymbol{r}_{O_ic_i} = \overline{O_ic_i}\,.
$$

Similarly, for the properties of the cross product, we have:

$$
{}^i\dot{\boldsymbol{X}}_{c_i/0}{}^T = {}^i\dot{\boldsymbol{X}}_{c_i/0}{}^T \boldsymbol{\phi}^*({}^i\boldsymbol{r}_{O_ic_i}) \tag{2.67}
$$

with:

$$
\boldsymbol{\phi}^*(\boldsymbol{r}) = \begin{bmatrix} \boldsymbol{I} & \left[\boldsymbol{r}\Lambda\right] \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}. \tag{2.68}
$$

It is then possible to express, recursively, the generalized velocity of the frame integral with the generic link *i* with the relation:

$$
{}^i\dot{\boldsymbol{X}}_{i/0} = \begin{bmatrix} {}^i_{i-1}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^i_{i-1}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}\left({}^{i-1}\boldsymbol{r}^{oj}_{i-1}\right) {}^{i-1}\dot{\boldsymbol{X}}_{i-1/0} + H_i\,\boldsymbol{p}_i \tag{2.69}
$$

where $\boldsymbol{r}^{oj}_{i-1}$ is the vector joining the origin of $<i-1>$ with the connection point of the joint i (fig. 2.8) and $\boldsymbol{p}_i$ is the associated quasivelocity vector. Note that the quantity $H_i\boldsymbol{p}_i$ gives according to the 2.25, the generalized velocity introduced by the joint and expressed in the "exit" frame[3].

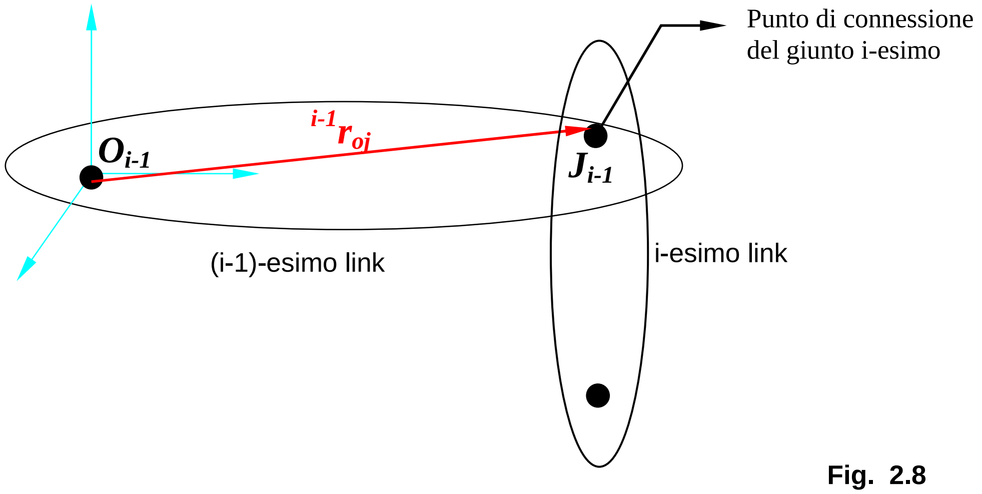

Fig. 2.8

Introducing the matrix

$$ {}^{i}_{i-1}\theta(\boldsymbol{r}) = \begin{bmatrix} {}^{i}_{i-1}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{i}_{i-1}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}(\boldsymbol{r}) \tag{2.70} $$

the 2.69 becomes:

$$ {}^{i}\dot{\boldsymbol{X}}_{i/0} = {}^{i}_{i-1}\theta\left({}^{i-1}\boldsymbol{r}^{oj}_{i-1}\right){}^{i-1}\dot{\boldsymbol{X}}_{i-1/0} + H_i\,\boldsymbol{p}_i \tag{2.71} $$

Note that ${}^{i}_{i-1}\theta(\boldsymbol{r})$ is the matrix that transforms the generalized velocity of the frame $<i-1>$ (expressed in the same frame $<i-1>$) into the generalized velocity of the point integral with the frame $<i-1>$ and specified by $\boldsymbol{r}$ (the last velocity is expressed in the frame $<i>$).

The 2.71 is already sufficient to recursively express all the different generalized velocities $\dot{X}_{i/0}$ of every frame of the robot. However, it is convenient to expand the recursive process into a unique global form. At this aim, let's introduce the following vectors:

---

[3] For example, with reference to the fig. 2.6, this quantity represents the generalized velocity of the frame <*b*> w.r.t. <*a*> expressed in the frame <*b*>.

$$\boldsymbol{V}_O = \begin{bmatrix} {}^1\dot{\boldsymbol{X}}_{1/0} \\ {}^2\dot{\boldsymbol{X}}_{2/0} \\ \vdots \\ {}^k\dot{\boldsymbol{X}}_{k/0} \end{bmatrix}, \quad \boldsymbol{p} = \begin{bmatrix} \boldsymbol{p}_1 \\ \boldsymbol{p}_2 \\ \vdots \\ \boldsymbol{p}_k \end{bmatrix}$$

and the matrix:

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{H}_1 & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{H}_2 & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{H}_k \end{bmatrix} \tag{2.72}$$

From the 2.71 it’s easy to verify that:

$$\boldsymbol{V}_O = \boldsymbol{\Phi}_l \, \boldsymbol{H} \, \boldsymbol{p} \tag{2.73}$$

where:

$$\boldsymbol{\Phi}_l = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ {}_1^2\theta & \boldsymbol{I} & \cdots & \boldsymbol{0} \\ \vdots & \vdots & \ddots & \boldsymbol{0} \\ {}_1^n\theta & {}_2^n\theta & \cdots & \boldsymbol{I} \end{bmatrix}, \quad {}_j^i\theta = {}_{i-1}^{\;\;i}\theta \cdots {}_j^{j+1}\theta, \quad i = 2 \ldots n, \, j = 1 \ldots n-1 \tag{2.74}$$

The matrix $\boldsymbol{\Phi}_l \boldsymbol{H}$ is the global jacobian of the robot; its dimension is $6n \times n$ and can be partitioned into k blocks such as:

$$\boldsymbol{V}_O = \begin{bmatrix} {}^1\dot{X}_{1/0} \\ {}^2\dot{X}_{2/0} \\ \vdots \\ {}^k\dot{X}_{k/0} \end{bmatrix} = \begin{bmatrix} {}^1J_{1/0}(O_1) \\ {}^2J_{2/0}(O_2) \\ \\ {}^kJ_{k/0}(O_k) \end{bmatrix} \boldsymbol{p} \tag{2.75}$$

where the matrix ${}^iJ_{j/0}(P)$ is the *jacobian*, that is:

$${}^i\dot{X}_{P/0} = {}^iJ_{j/0}(P) \, \boldsymbol{p} \tag{2.76}$$

where P is a generic point integral with the frame $< j >$.

2.4.1 *Estension to open branched chains*

The matrix $\boldsymbol{\Phi}_l$ of 2.74 has been computed in case of linear chains.

Let’s now extend its computation in presence of a kinematic chain with a branched topology.

Let’s consider a generic structure described as in paragraph 1.4 and having B branches, each branch i made of a linear sequence of k_i links.

It is possible to identify each link with the index pair (i, j), with $1 \le i \le B$ and $1 \le j \le k_i$. The joint associated to such link will have the following specifications:

- $r_{i,j}$ degrees of freedom;
- joint matrix $H_{i,j} \in \Re^{6 \times r_{i,j}}$;
- configuration vector $\boldsymbol{q}_{i,j} \in \Re^{r_{i,j}}$
- quasivelocity vector $\boldsymbol{p}_{i,j} \in \Re^{r_{i,j}}$

The *global configuration vector* is then defined as:

$$\boldsymbol{q} \hat{=} \left[\boldsymbol{q}_{1,1}{}^T \quad \cdots \quad \boldsymbol{q}_{1,k_1}{}^T \quad \boldsymbol{q}_{2,1}{}^T \quad \cdots \quad \boldsymbol{q}_{2,k_2}{}^T \quad \cdots \quad \boldsymbol{q}_{B,1}{}^T \quad \cdots \quad \boldsymbol{q}_{B,k_B}{}^T \right]^T \qquad (2.77)$$

Similarly, the *global quasivelocity vector* of the structure is defined as:

$$\boldsymbol{p} \hat{=} \left[\boldsymbol{p}_{1,1}{}^T \quad \cdots \quad \boldsymbol{p}_{1,k_1}{}^T \quad \boldsymbol{p}_{2,1}{}^T \quad \cdots \quad \boldsymbol{p}_{2,k_2}{}^T \quad \cdots \quad \boldsymbol{p}_{B,1}{}^T \quad \cdots \quad \boldsymbol{p}_{B,k_B}{}^T \right]^T \qquad (2.78)$$

Finally, defining

$$H \hat{=} \operatorname{diag}\left[H_{1,1}{}^T \quad \cdots \quad H_{1,k_1}{}^T \quad H_{2,1}{}^T \quad \cdots \quad H_{2,k_2}{}^T \quad \cdots \quad H_{B,1}{}^T \quad \cdots \quad H_{B,k_B}{}^T \right]^T$$

(2.79)

$$\boldsymbol{V}_0 \hat{=} \begin{bmatrix} {}^{(1,1)}\dot{X}_{(1,1)/0} \\ \vdots \\ {}^{(1,k_1)}\dot{X}_{(1,k_1)/0} \\ {}^{(2,1)}\dot{X}_{(2,1)/0} \\ \vdots \\ {}^{(2,k_2)}\dot{X}_{(2,k_2)/0} \\ \vdots \\ {}^{(B,1)}\dot{X}_{(B,1)/0} \\ \vdots \\ {}^{(B,k_B)}\dot{X}_{(B,k_B)/0} \end{bmatrix} \qquad (2.80)$$

we want to find a relation similar to the 2.73, that is:

$$\boldsymbol{V}_O = \boldsymbol{\Phi}\,\boldsymbol{H}\,\boldsymbol{p} \tag{2.81}$$

The recursive formula 2.69 is still valid, provided that it will be evaluated on a path (unique) that, starting from the link in consideration, reaches the base frame. For example, lets compute the generalized velocity of the frame $< 2,3 >$ of figure 2.9.

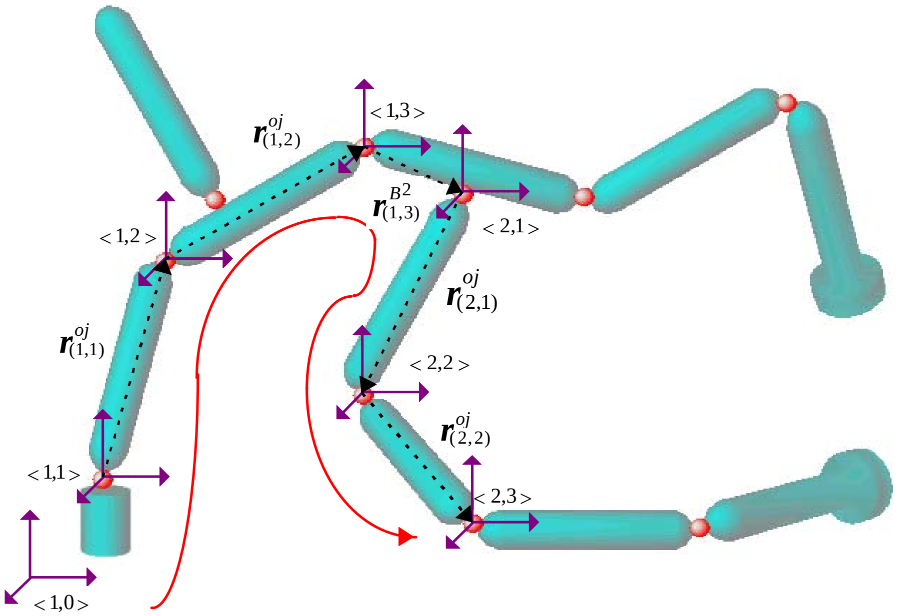

Fig. 2.9

We obtain the following recursion:

$${}^{(1,1)}\dot{X}_{(1,1)/(1,0)} = H_{(1,1)}\,\boldsymbol{p}_{(1,1)}$$

$${}^{(1,2)}\dot{\boldsymbol{X}}_{(1,2)/(1,0)} = \begin{bmatrix} {}^{(1,2)}_{(1,1)}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(1,2)}_{(1,1)}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}\left({}^{(1,1)}\boldsymbol{r}^{oj}_{(1,1)}\right){}^{(1,1)}\dot{\boldsymbol{X}}_{(1,1)/(1,0)} + H_{(1,2)}\,\boldsymbol{p}_{(1,2)}$$

$${}^{(1,3)}\dot{\boldsymbol{X}}_{(1,3)/(1,0)} = \begin{bmatrix} {}^{(1,3)}_{(1,2)}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(1,3)}_{(1,2)}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}\left({}^{(2,2)}\boldsymbol{r}^{oj}_{(2,2)}\right){}^{(1,2)}\dot{\boldsymbol{X}}_{(1,2)/(1,0)} + H_{(1,3)}\,\boldsymbol{p}_{(1,3)}$$

$$
\begin{aligned}
{}^{(2,1)}\dot{\boldsymbol{X}}_{(2,1)/(1,0)} &= \begin{bmatrix} {}^{(2,1)}_{(1,3)}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(2,1)}_{(1,3)}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}\left({}^{(1,3)}\boldsymbol{r}^{B2}_{(1,3)}\right) {}^{(1,3)}\dot{\boldsymbol{X}}_{(1,3)/(1,0)} + H_{(2,1)}\boldsymbol{p}_{(2,1)} \\
{}^{(2,2)}\dot{\boldsymbol{X}}_{(2,2)/(1,0)} &= \begin{bmatrix} {}^{(2,2)}_{(2,1)}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(2,2)}_{(2,1)}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}\left({}^{(2,1)}\boldsymbol{r}^{oj}_{(2,1)}\right) {}^{(2,1)}\dot{\boldsymbol{X}}_{(2,1)/(1,0)} + H_{(2,2)}\boldsymbol{p}_{(2,2)} \\
{}^{(2,3)}\dot{\boldsymbol{X}}_{(2,3)/(1,0)} &= \begin{bmatrix} {}^{(2,3)}_{(2,2)}\boldsymbol{R} & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(2,3)}_{(2,2)}\boldsymbol{R} \end{bmatrix} \boldsymbol{\phi}\left({}^{(2,2)}\boldsymbol{r}^{oj}_{(2,2)}\right) {}^{(2,2)}\dot{\boldsymbol{X}}_{(2,2)/(1,0)} + H_{(2,3)}\boldsymbol{p}_{(2,3)}
\end{aligned}
\tag{2.82}
$$

Even in this case it is convenient to express the whole process in a global form. Let the matrix Φ be the following:

$$
\Phi =
$$

$$
\left[\begin{array}{cccc|cccc|c|cccc}
\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
{}^{(1,2)}_{(1,1)}\boldsymbol{\psi} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
{}^{(1,k_1)}_{(1,1)}\boldsymbol{\psi} & {}^{(1,k_1)}_{(1,2)}\boldsymbol{\psi} & \cdots & \boldsymbol{I} & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
\hline
{}^{(2,1)}_{(1,1)}\boldsymbol{\psi} & {}^{(2,1)}_{(1,2)}\boldsymbol{\psi} & \cdots & {}^{(2,1)}_{(1,k_1)}\boldsymbol{\psi} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
{}^{(2,2)}_{(1,1)}\boldsymbol{\psi} & {}^{(2,2)}_{(1,2)}\boldsymbol{\psi} & \cdots & {}^{(2,2)}_{(1,k_1)}\boldsymbol{\psi} & {}^{(2,2)}_{(2,1)}\boldsymbol{\psi} & \boldsymbol{I} & \cdots & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
{}^{(2,k_2)}_{(1,1)}\boldsymbol{\psi} & {}^{(2,k_2)}_{(1,2)}\boldsymbol{\psi} & \cdots & {}^{(2,k_2)}_{(1,k_1)}\boldsymbol{\psi} & {}^{(2,k_2)}_{(2,1)}\boldsymbol{\psi} & {}^{(2,k_2)}_{(2,2)}\boldsymbol{\psi} & \cdots & \boldsymbol{I} & \cdots & \boldsymbol{0} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
\hline
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots \\
\hline
{}^{(B,1)}_{(1,1)}\boldsymbol{\psi} & {}^{(B,1)}_{(1,2)}\boldsymbol{\psi} & \cdots & {}^{(B,1)}_{(1,k_1)}\boldsymbol{\psi} & {}^{(B,1)}_{(2,1)}\boldsymbol{\psi} & {}^{(B,1)}_{(2,2)}\boldsymbol{\psi} & \cdots & {}^{(B,1)}_{(2,k_2)}\boldsymbol{\psi} & \cdots & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\
{}^{(B,2)}_{(1,1)}\boldsymbol{\psi} & {}^{(B,2)}_{(1,2)}\boldsymbol{\psi} & \cdots & {}^{(B,2)}_{(1,k_1)}\boldsymbol{\psi} & {}^{(B,2)}_{(2,1)}\boldsymbol{\psi} & {}^{(B,2)}_{(2,2)}\boldsymbol{\psi} & \cdots & {}^{(B,2)}_{(2,k_2)}\boldsymbol{\psi} & \cdots & {}^{(B,2)}_{(B,1)}\boldsymbol{\psi} & \boldsymbol{I} & \cdots & \boldsymbol{0} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
{}^{(B,k_B)}_{(1,1)}\boldsymbol{\psi} & {}^{(B,k_B)}_{(1,2)}\boldsymbol{\psi} & \cdots & {}^{(B,k_B)}_{(1,k_1)}\boldsymbol{\psi} & {}^{(B,k_B)}_{(2,1)}\boldsymbol{\psi} & {}^{(B,k_B)}_{(2,2)}\boldsymbol{\psi} & \cdots & {}^{(B,k_B)}_{(2,k_2)}\boldsymbol{\psi} & \cdots & {}^{(B,k_B)}_{(B,1)}\boldsymbol{\psi} & {}^{(B,k_B)}_{(B,2)}\boldsymbol{\psi} & \cdots & \boldsymbol{I}
\end{array}\right]
\tag{2.83}
$$

Despite the apparent complexity due to a massive presence of indexes, the matrix ${}^{(r,s)}_{(i,j)}\boldsymbol{\psi}$ can be obtained through a simple recursive algorithm, obtained extrapolating the previous example. We have:

$$
{}^{(r,s)}_{(i,j)}\boldsymbol{\psi} = \begin{cases} \boldsymbol{0}^{6\times 6} & r<i \\ \boldsymbol{0}^{6\times 6} & s<j \quad r=i \\ \boldsymbol{I}^{6\times 6} & s=j \quad r=i \\ \begin{bmatrix} {}^{(r,s)}_{(r,s-1)}R & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(r,s)}_{(r,s-1)}R \end{bmatrix} \boldsymbol{\phi}\left({}^{(r,s-1)}\boldsymbol{r}^{oj}_{(r,s-1)}\right) {}^{(r,s)}_{(i,s-1)}\boldsymbol{\psi} & s>j \quad r=i \\ \begin{bmatrix} {}^{(r,1)}_{(B_r,L_r)}R & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(r,1)}_{(B_r,L_r)}R \end{bmatrix} \boldsymbol{\phi}\left({}^{(B_r,L_r)}\boldsymbol{r}^{Br}_{(B_r,L_r)}\right) {}^{(r,1)}_{(B_r,L_r)}\boldsymbol{\psi} & s=1 \quad r>i \\ \begin{bmatrix} {}^{(r,s)}_{(r,s-1)}R & \boldsymbol{0} \\ \boldsymbol{0} & {}^{(r,s)}_{(r,s-1)}R \end{bmatrix} \boldsymbol{\phi}\left({}^{(r,s-1)}\boldsymbol{r}^{oj}_{(r,s-1)}\right) {}^{(r,s)}_{(r,s-1)}\boldsymbol{\psi} & s>1 \quad r>i \end{cases} \tag{2.84}
$$

where, as usual, ${}^{(r,s-1)}\boldsymbol{r}^{oj}_{(r,s-1)}$ is the point of connection of the joint s of the branch r expressed in the frame $<r,s-1>$, while ${}^{(B_r,L_r)}\boldsymbol{r}^{Br}_{(B_r,L_r)}$ is the connection point of the joint 1 of the branch r expressed in the frame $<B_r,L_r>$ (the branch r is attached to the link L_r, belonging to the branch B_r).

In this way the 2.81 holds and, similarly to the case of linear chains, the product $\boldsymbol{\Phi H}$ is the global jacobian of the robot. It can be partitioned into k blocks such as:

$$
\boldsymbol{V}_O = \begin{bmatrix} {}^{(1,1)}\dot{X}_{(1,1)/(1,0)} \\ \vdots \\ {}^{(1,k_1)}\dot{X}_{(1,k_1)/(1,0)} \\ {}^{(2,1)}\dot{X}_{(2,1)/(1,0)} \\ \vdots \\ {}^{(2,k_2)}\dot{X}_{(2,k_2)/(1,0)} \\ \vdots \\ {}^{(B,1)}\dot{X}_{(B,1)/(1,0)} \\ \vdots \\ {}^{(B,k_B)}\dot{X}_{(B,k_B)/(1,0)} \end{bmatrix} = \begin{bmatrix} {}^{(1,1)}J_{(1,1)/(1,0)} \\ \vdots \\ {}^{(1,k_1)}J_{(1,k_1)/(1,0)} \\ {}^{(2,1)}J_{(2,1)/(1,0)} \\ \vdots \\ {}^{(2,k_2)}J_{(2,k_2)/(1,0)} \\ \vdots \\ {}^{(B,1)}J_{(B,1)/(1,0)} \\ \vdots \\ {}^{(B,k_B)}J_{(B,k_B)/(1,0)} \end{bmatrix} \boldsymbol{p} \tag{2.85}
$$

where the matrix ${}^{(r,s)}J_{(i,j)/(1,0)}(P)$ is the *jacobian* of the structure and is such as:

$$
{}^{(r,s)}\dot{X}_{(i,j)/(1,0)} = {}^{(r,s)}J_{(i,j)/(1,0)}(P)\,\boldsymbol{p} \tag{2.86}
$$

being P a generic point integral to the frame $< i, j >$.